(Grove, 1988)

This paper proposes a model-theoretic characterization of the belief revision ($*$) operation discussed by AGM. Grove's idea is to give a “spheres semantics” inspired by (Lewis, 1973a) for $*$, thus enriching the purely syntactic approach by AGM.

Grove's sets the stage without taking a stance on some (trivial) background assumptions, so I will fill in the gaps.

Let F be the set of well-formed sentences in a Boolean language, and $L \subseteq F$ be the set of “logical theses of some logic”, i.e. let L be the set of propositional tautologies and to be closed under modus ponens. Define:

1. A theory $T \in \mathcal{T}$ is a subset of F that is closed under modus ponens, i.e. $Cn(T) = T$.
2. A set of formulas S is consistent, $Con(S)$, iff $Con(S) \neq F$.
3. The revision function $+ : \mathcal{T} \times F \to \mathcal{T}$ satisfies the following axioms: for any $T \in \mathcal{T}$ and $A \in F$,
   - (+1) $T + A = Cn(T + A)$ [this holds trivially],
   - (+2) $A \in T + A$,
   - (+3) $T + A = Cn(T \cup \{A\})$, if $\neg A \notin T$,
   - (+4) $Cn(T + A) \neq F$, if $\neg A \notin L$,
   - (+5) $T + A = T + B$, if $A \leftrightarrow B \in L$,
   - (+7) $T + (A \wedge B) \subseteq Cn((T + A) \cup \{B\})$,
   - (+8) $Cn((T + A) \cup \{B\}) \subseteq T + (A \wedge B)$, if $\neg B \notin T + A$.

There are two objects that deserve extensive discussion: maximal consistent extensions of L, and the function t.

1. Maximal Consistent Extensions of L

Grove does not take the notion of possible world as primitive, but he starts from syntactic objects (sentences) and then constructs other objects (maximal consistent sets of sentences) that are basically equivalent to possible worlds.

The set M_L is the set of maximal consistent extensions of L, and it consists of sets $m \subseteq F$ such that:

Definition 1 (Maximal Consistent Extensions of the Logic L).

Let $L \subseteq F$ be a logic, where F is the set of all well-formed formulae. Then, define M_L as the set of consistent maximal extensions of the logic L, that is

$$m \in M_L \iff \begin{cases} 1.\ L \subseteq m & \\ 2.\ Cn(m) \neq F & \text{(Consistency)} \\ 3.\ \forall n \subseteq F,\ \text{if } m \subset n,\ \text{then } Cn(n) = F & \text{(Maximality)} \end{cases}$$

Note that, for any $m \in M_L$, m is a theory.

Proposition 2.

For any $m \in M_L$, m is a theory.

Proof. Consider $m \in M_L$ and suppose for reductio that $m \neq Cn(m)$. Since $m \subseteq Cn(m)$ by the Reflexivity of Cn, $m \neq Cn(m)$ implies that there is $x \in Cn(m)$ such that $x \notin m$. Set $n := Cn(m)$. Clearly, we have $n \in \mathcal{T}$ and $m \subset n$. Since $m \in M_L$, m must be maximal, and hence $n = Cn(m) = F$, contradicting the consistency of m. Therefore, $m = Cn(m)$. □

The following propositions clarifies why m is a possible world. First, let me establish that M_L could have been defined in a different, equivalent way.

Proposition 3.

Let $m \subseteq F$.

$$m \in M_L \iff \begin{cases} 1.\ L \subseteq m & \\ 2.\ Cn(m) \neq F & \text{(Consistency)} \\ 3.\ \text{ For all } A \in F\text{: either } A \in m \text{ or } \neg A \in m. & \text{(Completeness)} \end{cases}$$

Proof. ($\Longrightarrow$) Take arbitrary $m \in M_L$ and $A \in F$. Obviously, (1) and (2) hold. So, let me prove that (3) holds as well. We have two cases: either $A \in m$ or $A \notin m$.

1. If $A \in m$ we are done.
2. Suppose that $A \notin m$. Consider now $n := Cn(m \cup \{A\})$. Since $m \subseteq m \cup \{A\}$, by Monotonicity $Cn(m) \subseteq Cn(m \cup \{A\})$, and by Reflexivity $m \subseteq Cn(m) \subseteq Cn(m \cup \{A\}) = n$. Since $A \in n$ but $A \notin m$, $m \subset n$. By the maximality of m, since $m \subset n$, it must be the case that $n = Cn(m \cup \{A\}) = F$. Therefore, $m \cup \{A\}$ must be inconsistent, meaning $m \cup \{A\} \vdash \bot$. By the deduction theorem for classical logic, we have that $m \vdash A \to \bot$. Since $A \to \bot$ is classically equivalent to $\neg A$, we have that $m \vdash \neg A$. Since m is a theory (i.e., $m = Cn(m)$), it follows that $\neg A \in m$.

Therefore, either $A \in m$ or $\neg A \in m$.

($\Longleftarrow$) Suppose that, $L \subseteq m$, $Cn(m) \neq F$, and that for all $A \in F$, either $A \in m$ or $\neg A \in m$. We need to prove that m is a theory and that it is maximal.

1. Let me first prove that m is a theory. We need to prove that $m = Cn(m)$. ($\subseteq$) Obviously $m \subseteq Cn(m)$. ($\supseteq$) Let us show that, if $A \in Cn(m)$, then $A \in m$. Suppose for reductio that $A \in Cn(m)$ and $A \notin m$. By the completeness of m, $\neg A \in m$. Since $m \subseteq Cn(m)$, $\neg A \in Cn(m)$. Since $A, \neg A \in Cn(m)$, $Cn(m) = F$, contradicting the consistency of m. Therefore, $A \in m$. Since A is arbitrary, we have $Cn(m) \subseteq m$, proving that m is a theory, i.e. $m = Cn(m)$.
2. Let me now prove that m is maximal. Suppose that and $m \subset n$, namely $m \subseteq n$ and there exists some $A \in n$ such that $A \notin m$. By the completeness of m, $\neg A \in m$. Since $m \subset n$, both A and $\neg A$ are in n. Hence, $Cn(n) = F$. Therefore, m is maximal.

Therefore, if m has (1)-(3) it must be maximal, and so $m \in M_L$. □

Proposition 4.

For any $m \in M_L$, there exists exactly one possible world $w \in W$ such that $w \models m$.

Proof. First, recall that a possible world w is basically a function (a valuation) from the set of propositional variables $\Phi \subseteq F$ that maps each $p_i \in \Phi$ to either 0 (false) or 1 (true). The satisfaction relation $w \models A$ for any complex formula $A \in F$ is defined recursively from this basis.

Consider now any $m \in M_L$.

Part 1: Existence. We must first construct a world w_m from m and show that it satisfies m.

First, let us construct the world w_m. We define a valuation $w_m : \Phi \to \{0, 1\}$ based on the contents of m. For any propositional variable $p \in \Phi$:

- If $p \in m$, we define $w_m(p) = 1$.
- If $p \notin m$, we define $w_m(p) = 0$. Note that, from Proposition 3, we know that if $p \notin m$, then $\neg p \in m$. So this definition is complete.

Second, let us show that $w_m \models m$: We must show that this w_m satisfies every formula in m. We can prove by induction on the structure of any formula $A \in F$ that $A \in m \iff w_m \models A$.

- **Base Case (Atoms):** For any $p \in \Phi$, $p \in m \iff w_m(p) = 1 \iff w_m \models p$. This holds by our definition of w_m.
- **Inductive Step (Negation):** Assume $B \in m \iff w_m \models B$. We show $\neg B \in m \iff w_m \models \neg B$.
  - $w_m \models \neg B \iff w_m \not\models B$ (by definition of $\models$)
  - $\iff B \notin m$ (by inductive hypothesis)
  - $\iff \neg B \in m$ (by Proposition 3).

- **Inductive Step (Conjunction):** Assume $B \in m \iff w_m \models B$ and $C \in m \iff w_m \models C$.
  - $w_m \models B \wedge C \iff w_m \models B$ and $w_m \models C$ (by definition of $\models$).
  - $\iff B \in m$ and $C \in m$ (by inductive hypothesis)
  - $\iff (B \wedge C) \in m$ (since m is a theory, it's closed under conjunction)

(This holds for all other connectives as well.) Since $A \in m \iff w_m \models A$ holds for all $A \in F$, it follows that m satisfies all and only the formulas A in m. Therefore, $w_m \models m$. This proves at least one such world exists.

Part 2: Uniqueness. We must now show that w_m is the only such world. Assume there is another world $w' \in W$ (a different valuation) such that $w' \models m$. To show $w' = w_m$, we only need to show that they agree on all propositional variables, i.e., $w'(p) = w_m(p)$ for all $p \in \Phi$. Let p be an arbitrary propositional variable in Φ.

- **Case 1:** $p \in m$**.**
  - By our construction, $w_m(p) = 1$.
  - Since $w' \models m$ and $p \in m$, it must be that $w' \models p$.
  - By definition of $\models$, this means $w'(p) = 1$.
  - Thus, $w'(p) = w_m(p)$.
- **Case 2:** $p \notin m$**.**
  - By our construction, $w_m(p) = 0$.
  - Since m is maximal and $p \notin m$, we know by Proposition 3 $\neg p \in m$.
  - Since $w' \models m$ and $\neg p \in m$, it must be that $w' \models \neg p$.
  - By definition of $\models$, this means $w'(p) = 0$.
  - Thus, $w'(p) = w_m(p)$.

Since w' and w_m agree on all propositional variables, they are the same valuation, i.e., $w' = w_m$. Therefore, for any $m \in M_L$, there exists one and only one world w that satisfies it. ◻

In other words, there exists a **function** $f : M_L \to W$ such that $f(m) = w_m$ such that $w \models m$. This function is also a **bijection**.

Proposition 5.

For all $w \in W$ there exists a unique $m \in M_L$ such that $w \models m$.

Proof. Consider an arbitrary $w \in W$.

Part 1: Existence. Construct the set m_w as follows

$$m_w := \{A \in F : w \models A\}$$

Clearly, it follows that $w \models m_w$ by construction—i.e., $w \models A$ for all $A \in m_w$. Second, let me prove that m_w is in M_L.

1. m_w **is a theory**. Obviously $m_w \subseteq Cn(m_w)$, so consider an arbitrary $A \in Cn(m_w)$ and let me show that $A \in m_w$. Since $m_w \vdash A$, by the Soundness of classical propositional logic, $m_w \models A$. Since $w \models m_w$, $w \models A$. Therefore, by the definition of m_w, $A \in m_w$. Since $Cn(m_w) \subseteq m_w$, we have that m_w is a theory.
2. m_w **is an extension of** L. For all $B \in L$, we have that B is satisfied by any world in W. So, $w \models B$ for all $B \in L$. Hence, $L \subseteq m_w$ for by construction.
3. m_w **is consistent**. Suppose for reductio that m_w is inconsistent, i.e. $Cn(m_w) = m_w = F$ (for m_w is a theory). Then, we have that $w \models F$, and hence $w \models \bot$, contradicting $w \not\models \bot$. Hence, m_w is consistent.
4. m_w **is maximal**. Suppose $m_w \subset n$ for some $n \subseteq F$. There must be some $A \in F$ such that $A \in n$ and $A \notin m_w$. Since $A \notin m_w$, it follows that $w \not\models A$, and hence by the definition of $\models$, $w \models \neg A$. By the construction of m_w, $\neg A \in m_w$. Since $m_w \subset n$, $\neg A \in n$. Since $\neg A, A \in n$, $Cn(n) = F$. Therefore, m_w is maximal.

Concluding, for any $w \in W$ there exists a set of formulas m_w such that $w \models m_w$ and m_w is a maximal consistent extension of L (i.e., $m \in M_L$).

Part 2: Uniqueness. Suppose $w \models m$ and $w \models m'$ with $m, m' \in M_L$. Suppose (without loss of generality) that there exists some A such that $A \in m$ and $A \notin m'$ ($m \neq m'$). By Proposition 3, since $A \notin m'$, $\neg A \in m'$. Therefore, since $w \models m$ and $w \models m'$, we have $w \models A$ and $w \models \neg A$. By the definition of $\models$, it follows that $w \models A$ and $w \not\models A$: contradiction. Therefore, $m = m'$. □

Before moving on, consider the following definition.

Definition 6.

Let $T \in \mathcal{T}$ be a theory. Define

$$|T| := \{m \in M_L : T \subseteq m\}.$$

In a way, $|T|$ is the set of possible worlds that make T true.

2. Properties of t

Define the following operation t.

Definition 7 (t).

Let $S \subseteq M_L$. Define the function $t : \mathcal{P}(M_L) \to \mathcal{P}(F)$ such that:

$$t(S) = \bigcap S.$$

Note that, if $S = \emptyset$, then $t(S) = F$. For $\bigcap S = \{x \in F : \text{For all } R \in S\ (x \in R)\}$, but if $S = \emptyset$ the condition—i.e. $\text{For all } R \in S\ (x \in R)\}$—is (trivially) satisfied by any $x \in F$.

Proposition 8 (Properties of t).

Let $S \subseteq M_L$.

1. *$t(S)$ is a theory.*
2. *$t(|T|) = T$ (for all theories T, assuming compactness).*
3. *$t(S)$ is consistent if and only if $S \neq \emptyset$.*
4. *$t(S \cap |A|) = Cn(t(S) \cup \{A\})$.*
5. *If $S \subseteq S'$, then $t(S') \subseteq t(S)$.*
6. *If $T \subseteq T'$, then $|T'| \subseteq |T|$.*

Let $S \subseteq M_L$ be a collection of maximal and consistent extensions of L.

(1) $t(S)$ is a theory. ($\subseteq$) Obviously, $t(S) \subseteq Cn(t(S))$ by the Reflexivity of Cn. ($\supseteq$) Let us show the other direction. Consider an arbitrary $x \in Cn(t(S))$. Recall that $t(S) = \bigcap S$.

1. Since $t(S) \subseteq m$ for every $m \in S$, by the Monotonicity of Cn, we have $Cn(t(S)) \subseteq Cn(m)$ for every $m \in S$.
2. Since every $m \in M_L$ is a theory (maximal consistent sets are closed), $Cn(m) = m$.
3. Therefore, $Cn(t(S)) \subseteq m$ for every $m \in S$.
4. Consequently, $x \in m$ for all $m \in S$.
5. Thus, $x \in \bigcap S = t(S)$.

(2) $t(|T|) = T$, if T is a theory and we have compactness. By definition, $|T| = \{m \in M_L : T \subseteq m\}$ and so $t(|T|) = \bigcap |T| = \bigcap \{m \in M_L : T \subseteq m\}$. Clearly, if T is inconsistent (i.e., $T = Cn(T) = F$), it follows that there is no m such that $T \subseteq m$ (if $T \subseteq m$, $m = F$ and hence $Cn(m) = F$, contradicting the consistency of m). Hence, $|T| = \emptyset$, so $t(|T|) = F$ and $T = F$ *ex hypothesi*, proving our equality. So, suppose that T is consistent.

($\supseteq$) Suppose $A \in T$. Since $T \subseteq m$ for all $m \in |T|$, $A \in m$ for all such m. Therefore $A \in \bigcap |T| = t(|T|)$.

($\subseteq$) Suppose $A \in t(|T|)$. Thus, $A \in m$ for all $m \in M_L$ such that $T \subseteq m$. Suppose for reductio that $A \notin T = Cn(T)$, for T is a theory. Since T is consistent *ex hypothesi* and $A \notin Cn(T)$, it follows that $T' := Cn(T \cup \{\neg A\}) \neq F$—if $Cn(T \cup \{\neg A\}) = F$, it follows that $T \vdash \neg A \to \bot$, i.e. $T \vdash A$, and so $A \in T$. Now, to prove a contradiction, we need to show that there exists a set m such that $T \subseteq T' \subseteq m$ and such that m is a maximal consistent extension of L (which implies $m \in |T|$ yet $A \notin m$). To do so, we need both Compactness and Zorn's Lemma.

1. First, define the collection of sets $\mathcal{K} := \{X \subseteq F : Cn(X) \neq F \text{ and } T' \subseteq X\}$. Clearly, $(\mathcal{K}, \subseteq)$ is a poset. Consider now an arbitrary chain $\mathcal{C} \subseteq \mathcal{K}$, and let us show that $\mathcal{C}$ has an upper bound in $\mathcal{K}$. Define $R := \bigcup_{X \in \mathcal{C}} X$. Clearly, $X \subseteq R$ for all $X \in \mathcal{C}$. We need to prove that $R \in \mathcal{K}$, i.e. that $T' \subseteq R$ and $Cn(R) \neq F$. Since $T' \subseteq X$ for all $X \in \mathcal{C}$, and $X \subseteq R$, it follows that $T' \subseteq R$. Suppose now for reductio that $Cn(R) = F$. By Compactness, there must be a finite subset $R_0 \subseteq R$ such that $Cn(R_0) = F$. Since $R_0 \subseteq R = \bigcup_{X \in \mathcal{C}} X$ is finite and $\mathcal{C}$ is a chain, there must be a set $X^* \in \mathcal{C}$ such that $R_0 \subseteq X^*$. By the Monotonicity of Cn, $Cn(R_0) \subseteq Cn(X^*)$, and so $Cn(X^*) = F$, contradicting the fact that $X^* \in \mathcal{K}$. Therefore, $Cn(R) \neq F$, and so $R \in \mathcal{K}$. Therefore, $\mathcal{C}$ has an upper bound in $\mathcal{K}$. Since $\mathcal{C}$ is an arbitrary chain in $\mathcal{K}$, we conclude that any such chain has an upper bound in $\mathcal{K}$.
2. By Zorn's Lemma, it follows that $\mathcal{K}$ has at least one maximal element: let us call it m. By definition of $\mathcal{K}$, $Cn(m) \neq F$ and $T' \subseteq m$. We must show that m is in M_L. We have already established that m is consistent. Note then that, since $T' := Cn(T \cup \{\neg A\})$, $L \subseteq T' \subseteq m$, hence m is an extension of L. Ultimately, note that m is maximal. If $m \subset n$, for some $n \subseteq F$, it must be the case that $Cn(n) = F$—if $Cn(n) \neq F$, then (since $T' \subseteq m \subset n$), $n \in \mathcal{K}$, contradicting the maximality of $m \in \mathcal{K}$. Thus, $m \in M_L$. Moreover, we have that $T \subseteq m$, for

$$T \quad \subseteq \quad T \cup \{\neg A\} \quad \subseteq \quad Cn(T \cup \{\neg A\}) \quad \subseteq \quad m$$

   Therefore, $m \in |T|$. However, $\neg A \in T' \subseteq m$. Since m is consistent, $A \notin m$. This contradicts the hypothesis that $A \in t(|T|)$ (which requires A to be in *every* $m \in |T|$). Therefore, we conclude that $A \in T$.
3. Since A is an arbitrary formula in $t(|T|)$, we have that $t(|T|) \subseteq T$.

We conclude that $t(|T|) = T$.

(3) $t(S)$ is consistent iff S is nonempty. ($\Longrightarrow$) If $S = \emptyset$, by definition we have that $t(S) = F$, and hence $Cn(t(S)) = F$. ($\Longleftarrow$) Suppose that $t(S)$ is inconsistent, meaning that $Cn(t(S)) = F$. Since $t(S) = \bigcap S$, where $S \subseteq M_L$, it clearly follows that $t(S) \subseteq m$ for all $m \in S$. If there is at least one $m \in S$, by the monotonicity of Cn, $Cn(t(S)) \subseteq Cn(m)$, implying that $Cn(m) = F$. However, this contradicts the consistency of m, thus there exists no $m \in S$, meaning $S = \emptyset$.

(4) $t(S \cap |A|) = Cn(t(S) \cup \{A\})$. We distinguish three cases based on the status of the set S.

Case 1: $S = \emptyset$**.** In this case, the intersection $S \cap |A| = \emptyset$. By definition, $t(\emptyset) = F$ (the inconsistent theory).

- **LHS:** $t(\emptyset) = F$.

- **RHS:** $Cn(t(\emptyset) \cup \{A\}) = Cn(F \cup \{A\}) = F$.
  The equality holds trivially.

Case 2: $S \neq \emptyset$, but $S \cap |A| = \emptyset$. In this case, the intersection is empty, so the LHS is $t(\emptyset) = F$. For the RHS, note that since $S \cap |A| = \emptyset$, it follows that for all $m \in S$, $m \notin |A|$, i.e., $A \notin m$. Since every $m \in M_L$ is complete, $\neg A \in m$ for all $m \in S$. Consequently, $\neg A \in \bigcap S = t(S)$. Therefore, the set $t(S) \cup \{A\}$ contains both $\neg A$ and A, making it inconsistent. Thus $Cn(t(S) \cup \{A\}) = F$. The equality holds.

Case 3: $S \cap |A| \neq \emptyset$. This is the principal case. Let us prove the equality by mutual inclusion.

($\subseteq$) Let $x \in t(S \cap |A|)$.

1. We prove that $A \to x \in t(S)$. Suppose for reductio that $A \to x \notin t(S)$. Since $t(S) = \bigcap S$, there must exist some $m \in S$ such that $A \to x \notin m$.
2. Since m is complete, $\neg(A \to x) \in m$. This is classically equivalent to $A \wedge \neg x \in m$, which implies $A \in m$ and $\neg x \in m$.
3. Since $m \in S$ and $A \in m$, it follows that $m \in S \cap |A|$.
4. However, by our initial hypothesis, $x \in t(S \cap |A|)$, which implies $x \in m'$ for *all* $m' \in S \cap |A|$. Therefore, $x \in m$. We now have $x \in m$ and $\neg x \in m$, contradicting the consistency of m.
5. Therefore, the assumption was false, and $A \to x \in t(S)$. By Modus Ponens, $Cn(t(S) \cup \{A\})$ contains x.

($\supseteq$) Let $x \in Cn(t(S) \cup \{A\})$.

1. By the deduction theorem (or definition of consequence), $t(S) \cup \{A\} \vdash x$, which implies $t(S) \vdash A \to x$.
2. Since $t(S)$ is a theory, $(A \to x) \in t(S)$. By definition of $t(S)$, for all $m \in S$, $(A \to x) \in m$.
3. Consider now any $m \in S \cap |A|$. For such m, we have $m \in S$ and $A \in m$.
4. Since $(A \to x) \in m$ and $A \in m$, it follows by closure of m that $x \in m$.
5. Since x is in every $m \in S \cap |A|$, $x \in t(S \cap |A|)$.

(5) For $S, S' \subseteq M_L$, if $S \subseteq S'$, then $t(S') \subseteq t(S)$. Let $S, S' \subseteq M_L$ and suppose $S \subseteq S'$. By definition, $t(S') = \bigcap S'$, which is the set of all formulae A such that $A \in m$ for all $m \in S'$. Consider any $A \in t(S')$. By definition, A is contained in every world in S'. Since $S \subseteq S'$, every world $m \in S$ is also a world in S'. Therefore, A must be contained in every world $m \in S$.
Consequently, $A \in \bigcap S = t(S)$. Since A was arbitrary, $t(S') \subseteq t(S)$. This relates to the general set-theoretic property that $X \subseteq Y \implies \bigcap Y \subseteq \bigcap X$ (Reverse-Inclusion Property).

(6) For $T, T' \in \mathcal{T}$, if $T \subseteq T'$ then $|T'| \subseteq |T|$. Let $T, T' \in \mathcal{T}$ and suppose $T \subseteq T'$. Consider any $m' \in |T'|$, that is any m' such that $T' \subseteq m'$. Clearly, $T \subseteq T' \subseteq m'$, hence $T \subseteq m'$, so $m' \in |T|$. So, $|T'| \subseteq |T|$.

The proof is complete. □

3. Systems of Spheres

Definition 9 (Systems of Spheres).

Let $\mathbf{S}$ be a collection of subsets of M_L, i.e. $\mathbf{S} \subseteq \mathcal{P}(M_L)$. $\mathbf{S}$ is a *system of spheres centered on* X ($\mathbf{S}_X$) for some subset $X \subseteq M_L$, if it satisfies the following conditions:

- **(S1: Nestedness)** $\mathbf{S}$ is totally ordered by $\subseteq$: if $U, V \in \mathbf{S}$, then $U \subseteq V$ or $V \subseteq U$.
- **(S2: Centering)** X is the $\subseteq$-minimum of $\mathbf{S}$: $X \in S$ and, for all $U \in \mathbf{S}$, $X \subseteq U$.
- **(S3: ?)** $M_L \in \mathbf{S}$.
- **(S4: Limit Assumption)** If $A \in F$ and $U \cap |A| \neq \emptyset$ for some $U \in \mathbf{S}$, then there exists a sphere $U^* \in \mathbf{S}$ such that: (1) $U^* \cap |A| \neq \emptyset$, and (2) $V \cap |A| \neq \emptyset$ implies $U \subseteq V$ for all $V \in \mathbf{S}$.

The main differences between Definition 9 (Systems of Spheres) and (Lewis, 1973a) > ^93cc74 are the following:

1. Grove's systems of spheres are not required to be closed under nonempty intersections and finite unions.
2. More significantly, Grove's systems of spheres are not centered on single worlds, but on sets of “worlds”.

We have that, for any $A \in F$,

1. If $|A| \cap \bigcup \mathbf{S} \neq \emptyset$, then S4 ensures that there exists some sphere in $\mathbf{S}$, call it $c(A)$, which intersects $|A|$ and is smaller than any other sphere with this property.
2. If $|A| \cap \bigcup \mathbf{S} = \emptyset$, then S3 implies that $|A| = \emptyset$, and we take $c(A) := M_L$.

Thus, given a sphere system $\mathbf{S}$, we can define a function $f_{\mathbf{S}} : F \to \mathcal{P}(M_L)$ such that

$$f_{\mathbf{S}}(A) := |A| \cap c(A)$$

(*N.b.* recall that $c(A)$ is a *sphere*.) In other words, $f_{\mathbf{S}}$ selects the closes worlds in M_L to X where A holds. Note that S4 (the Limit Assumption) plays a crucial role here, for it ensures that $f_{\mathbf{S}}$ is nonempty when A is *not* inconsistent (i.e., $\neg A \notin L$).

Grove proposes a different kind of model, which he claims to be equivalent to Definition 9 (Systems of Spheres).

Definition 10 (Total Order on M_L).

Let $X \subseteq M_L$ and $\leq_X$ ($\leq$ for simplicity) be a relation on M_L satisfying the following:

- (≤ 1) $\leq$ is connected: for all $x, y \in M_L$, either $x \leq y$ or $y \leq x$.
- (≤ 2) $\leq$ is transitive.
- (≤ 3) If $|A| \neq \emptyset$, then the set $\{x \in |A| : x \leq y, \text{ for all } y \in |A|\} \neq \emptyset$ – this is the analogue of S4.
- (≤ 4) $x \in M_L$ is $\leq$-minimal (i.e., $x \leq y$ for all $y \in M_L$) if, and only if, $x \in X$.

Proposition 11.

Definition 9 (Systems of Spheres) and Definition 10 (Total Order on M_L) are equivalent.

Proof. TBD □

Before introducing the main proofs by Grove, consider the following useful lemma.

Lemma 12.

Let $A, B \in F$.

$$\begin{array}{ll} (1) & |A \wedge B| = |A| \cap |B| \\ (2) & |A \vee B| = |A| \cup |B| \end{array}$$

Proof. The proof is straightforward.

1. Note that $|A \wedge B|$ is the set of $m \in M_L$ such that $A \wedge B \in m$. Since m is a theory Proposition 2, $A, B \in m$, so $m \in |A| \cap |B|$. The other direction follows analogously.
2. Note that $|A \vee B|$ is the set of $m \in M_L$ such that $A \vee B \in m$. Since m is a theory Proposition 2, either $A \in m$ or $B \in m$, so $m \in |A| \cup |B|$. The other direction follows analogously.

This completes the proof. □

⚠ Augmentation ("$/$") *vs.* Revision ("$+$")

In Theorem 13 (Soundness), $T + A := t(f_S(A))$. It is useful to see what augmentation amounts to in terms of spheres system. First, by Proposition 8 (Properties of t), for any $S \subseteq M_L$ and $A \in F$,

$$t(S \cap |A|) = Cn(t(S) \cup \{A\})$$

Therefore, by properties (2) and (4) in Proposition 8 (Properties of t),

$$\begin{aligned} T/A &= Cn(T \cup \{A\}) \\ &= Cn(t(|T|) \cup \{A\}) \\ &= t(|T| \cap |A|) \end{aligned}$$

So, while *revising a theory T by A* means taking $t(f_{\mathbf{S}}(A))$, where $\mathbf{S}$ is a system centered on T, *expanding a theory T by A* means taking $t(|T| \cap |A|)$. (*N.b.*, if $|T| \cap |A| = \emptyset$, it follows that $t(|T| \cap |A|) = F$, as expected.)

4. Soundness and Completeness

The following two theorems are the main result of (Grove, 1988).

4.1. Soundness

Theorem 13 (Soundness).

Let $\mathbf{S}$ be a system of spheres in M_L centered on $|T|$ for some theory $T \in \mathcal{T}$. If we define, for any $A \in F$,

$$T + A := t(f_{\mathbf{S}}(A))$$

then the axioms (+2) to (+8) are all satisfied.

Proof. Let $\mathbf{S}$ be a system of spheres in M_L centered on $|T|$ for some theory $T \in \mathcal{T}$.

(+1) Clearly, $Cn(t(f_{\mathbf{S}}(A))) = t(f_{\mathbf{S}}(A))$, for $t(S)$ is a theory for any $S \subseteq M_L$ by Proposition 8 (Properties of t), and $f_{\mathbf{S}}(A) = c(A) \cap |A| \subseteq M_L$.

(+2) Let me prove that $A \in T + A$. We have $f_{\mathbf{S}}(A)) = t(|A| \cap c(A)$ by definition. Since $c(A)$ is a sphere, it is a set of $m \in M_L$. By Proposition 8 (Properties of t) (prop. 4), it follows that

$$\begin{aligned} t(f_{\mathbf{S}}(A)) &= t(|A| \cap c(A)) \\ &= Cn(t(c(A)) \cup \{A\}) \end{aligned}$$

and since $A \vdash A$, $A \in Cn(t(c(A)) \cup \{A\}) = t(f_{\mathbf{S}}(A))$.

(+3) Suppose $\neg A \notin T$, and let me prove that $T + A = Cn(T \cup \{A\})$. Since $\neg A \notin T$, it follows that $T \cup \{A\}$ is consistent, i.e. there exists some $m \in M_L$ such that $T \subseteq m$ and $A \in m$. Therefore, $|T| \cap |A| \neq \emptyset$. Since $|T|$ is the smallest sphere in $\mathbf{S}$ and $|T| \cap |A| \neq \emptyset$, it follows that $c(A) = |T|$. So,

$$t(f_{\mathbf{S}}(A)) = t(|A| \cap |T|)$$

and since $|T| \subseteq M_L$, it follows from Proposition 8 (Properties of t) (prop. 4)

$$\begin{aligned} t(f_{\mathbf{S}}(A)) &= t(|A| \cap |T|) \\ &= Cn(t(|T|) \cup \{A\}) \\ &= Cn(T \cup \{A\}) \qquad \text{[By Prop. 2 of } t] \end{aligned}$$

(+4) Suppose $\neg A \notin L$, and let me show that $Con(T + A)$, i.e. that $Cn(T + A) \neq F$. Let me show that $T + A \neq \emptyset$, and conclude from Proposition 8 (Properties of t) (prop. 3).

Since $\neg A \notin L$, A is not a contradiction, therefore there exists some $m \in M_L$ such that $A \in m$. So $|A|$ is nonempty. Since we also have that $|A| \cap \bigcup \mathbf{S} = M_L$ is nonempty, by S4 $c(A)$ is nonempty, and it is the smallest $U \in \mathbf{S}$ such that $U \cap |A|$ is nonempty. Hence, $|A| \cap c(A)$ is nonempty, and by Proposition 8 (Properties of t) (prop. 3), $t(|A| \cap c(A)) = t(f_{\mathbf{S}}(A))$ is consistent.

(+5) Suppose $A \leftrightarrow B \in L$, and let me show that $T + A = T + B$. Since $L \subseteq m$, for all $m \in M_L$, we have that $|A| = |B|$: if $m \in |A|$, since m is a theory and $A \leftrightarrow B \in m$, $m \in |B|$, and the same goes for any $m \in |B|$. Since $|A| = |B|$, $c(A) = c(B)$, and therefore $|A| \cap c(A) = |B| \cap c(B)$, which implies that

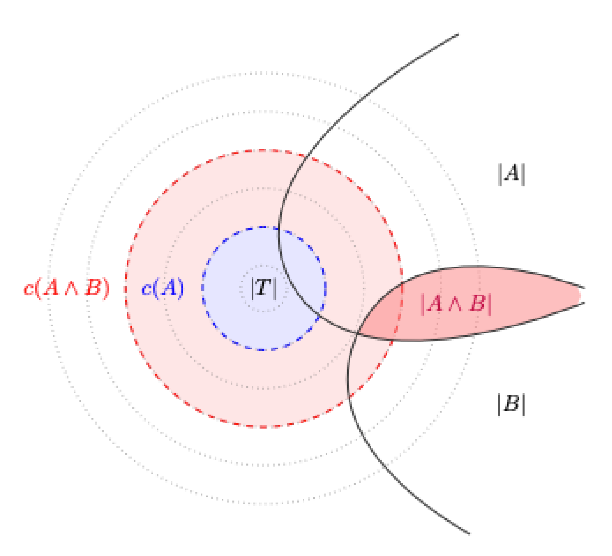
$$\begin{aligned} |A| \cap c(A) &= |B| \cap c(B) \\ f_{\mathbf{S}}(A) &= f_{\mathbf{S}}(B) \\ t(f_{\mathbf{S}}(A)) &= t(f_{\mathbf{S}}(B)) \\ T + A &= T + B \end{aligned}$$

(+7) We need to prove $T + (A \wedge B) \subseteq Cn((T + A) \cup \{B\})$. First, note that by Lemma 12, $|A \wedge B| = |A| \cap |B| \subseteq |A|$. This means that any sphere intersecting $|A \wedge B|$ does intersect $|A|$ as well. Consider now $c(A)$, i.e. the sphere U such that (1) $U \cap |A| \neq \emptyset$, and (2) $U \subseteq V$ for all V with $V \cap |A| \neq \emptyset$. Since $c(A \wedge B)$ intersects $|A \wedge B|$ by definition, from the reasoning above it follows that it does intersect $|A|$ as well. By the definition of $c(A)$, then $c(A) \subseteq c(A \wedge B)$.

For any $A, B \in F$: $c(A) \subseteq c(A \wedge B)$.

$c(A)$ is a sphere $U \in \mathbf{S}$ such that: $U \cap |A| \neq \emptyset$ and, for all $V \cap |A| \neq \emptyset$, $U \subseteq V$. Consider now $c(A \wedge B)$. Since $c(A \wedge B) \cap |A \wedge B| \neq \emptyset$ by definition, $c(A \wedge B) \cap |A| \neq \emptyset$. By the definition of $c(A)$, it follows that $c(A) \subseteq c(A \wedge B)$.

Here is a graph showing why this is the case:



Therefore,

$$\begin{aligned} c(A) &\subseteq c(A \wedge B) \\ c(A) \cap |A| \cap |B| &\subseteq c(A \wedge B) \cap |A| \cap |B| \end{aligned}$$

By prop. 5 of Proposition 8 (Properties of t) (i.e., the Reverse-Inclusion Property), it follows that

$$t(c(A \wedge B) \cap |A| \cap |B|) \subseteq t(c(A) \cap |A| \cap |B|)$$
$$t(c(A \wedge B) \cap |A \wedge B|) \subseteq t(c(A) \cap |A| \cap |B|)$$

By definition, $t(c(A \wedge B) \cap |A \wedge B|) = T + (A \wedge B)$. Regarding the RHS, apply prop. 4 of Proposition 8 (Properties of t) by setting $S := c(A) \cap |A| \subseteq M_L$, and so

$$\begin{aligned} t(c(A) \cap |A| \cap |B|) &= t\Big(\big(c(A) \cap |A|\big) \cap |B|\Big) \\ &= Cn\Big(t\big(c(A) \cap |A|\big) \cup \{B\}\Big) \\ &= Cn\Big(\big(T + A\big) \cup \{B\}\Big) \\ &= (T + A)/B \end{aligned}$$

Therefore, we conclude that $T + (A \wedge B) \subseteq (T + A)/B$.

(+8) Suppose $\neg B \notin T + A$. We need to prove $Cn((T + A) \cup \{B\}) \subseteq T + (A \wedge B)$. Since $\neg B \notin T + A$ and $T + A$ is a theory—see the proof of (+1)—it follows that B is consistent with the theory $T + A$, i.e. there exists at least one possible world $m \in M_L$ such that $m \in |B|$ and $m \in |T + A|$. Hence $|T + A| \cap |B| \neq \emptyset$. We need to prove that also the "generating set" of $T + A$, i.e. $c(A) \cap |A|$, is consistent with B, i.e. that $c(A) \cap |A| \cap |B| \neq \emptyset$.

Suppose for reductio that $c(A) \cap |A| \cap |B| = \emptyset$. So, for any $m \in c(A) \cap |A|$, $B \notin m$. Since m is complete for all $m \in M_L$, it follows that $\neg B \in m$ for all $m \in c(A) \cap |A|$. Thus, $\neg B \in t(c(A) \cap |A|) = T + A$, contradicting our hypothesis. Therefore,

$$c(A) \cap |A| \cap |B| = c(A) \cap |A \wedge B|$$
$$\neq \emptyset$$

Therefore, the closest sphere intersecting $|A|$ also intersects $|A \wedge B|$, and so by the definition of $c(A \wedge B)$, it follows that $c(A \wedge B) \subseteq c(A)$. [Also, consider that the definition of $c(A)$ and Lemma 12 imply that $c(A) \subseteq c(A \wedge B)$. Therefore, $c(A) = c(A \wedge B)$.]

We then argue as above:

$$\begin{array}{rcl} c(A \wedge B) & \subseteq & c(A) \\ c(A \wedge B) \cap |A| \cap |B| & \subseteq & c(A) \cap |A| \cap |B| \\ t(c(A) \cap |A| \cap |B|) & \subseteq & t(c(A \wedge B) \cap |A| \cap |B|) \end{array}$$

By definition, $t(c(A \wedge B) \cap |A \wedge B|) = T + (A \wedge B)$. Regarding the LHS, apply prop. 4 of Proposition 8 (Properties of t) by setting $S := c(A) \cap |A| \subseteq M_L$, and so

$$\begin{aligned} t(c(A) \cap |A| \cap |B|) &= t\Big(\big(c(A) \cap |A|\big) \cap |B|\Big) \\ &= Cn\Big(t\big(c(A) \cap |A|\big) \cup \{B\}\Big) \\ &= Cn\Big(\big(T + A\big) \cup \{B\}\Big) \\ &= (T + A)/B \end{aligned}$$

Therefore, we conclude that $(T + A)/B \subseteq T + (A \wedge B)$.

So, we have proved that, if $T + A := t(f_{\mathbf{S}}(A))$, then $+$ satisfies (+1)-(+8). □

4.2. Completeness

Theorem 14 (Completeness)**.**

Let $+ : \mathcal{T} \times F \rightarrow \mathcal{T}$ be any function satisfying (+1)-(+8). Then, for any fixed theory T, there is a system of spheres on M_L, $\mathbf{S}$, centered on $|T|$ and satisfying $T + A = t(f_{\mathbf{S}}(A))$ for all $A \in F$.

Proof. Let $+ : \mathcal{T} \times F \rightarrow \mathcal{T}$ be any function satisfying (+1)-(+8). Consider an arbitrary theory $T \in \mathcal{T}$. Define $\mathbf{S}'$ as the class of all **nonempty** subsets $U \subseteq M_L$ satisfying the following conditions:

1. (**Relevance**) For all $u \in U$ there is $A \in F$ such that $u \in |T + A|$, and
2. (**Swallowing**) For all $A \in F$, if $|A| \cap U \neq \emptyset$, then $|T + A| \subseteq U$.

Then, define $\mathbf{S}$ as follows:

$$\mathbf{S} := \begin{cases} \mathbf{S}' \cup \{M_L\} & T \text{ consistent} \\ \mathbf{S}' \cup \{M_L, \emptyset\} & \text{otherwise} \end{cases}$$

(In other words, the candidate sphere system $\mathbf{S}$ is obtained by adding the universe (and possibly the empty set) to $\mathbf{S}'$.) Let me now prove that $\mathbf{S}$ is a system of spheres.

Part One: $\mathbb{S}$ is a system of spheres.

(S1) Distinguish two cases.

Case 1: T is inconsistent. By construction, $\mathbf{S} = \mathbf{S}' \cup \{M_L, \emptyset\}$. Let $U, V \in \mathbf{S}$.

- If $U = \emptyset$ or $V = \emptyset$: Since $\emptyset \subseteq X$ for any set X, the nesting condition ($U \subseteq V$ or $V \subseteq U$) holds trivially.
- If $U = M_L$ or $V = M_L$: Since $X \subseteq M_L$ for any set of worlds X, nesting holds trivially.
- If $U, V \in \mathbf{S}'$: These are nonempty sets satisfying Relevance and Swallowing. The proof is identical to **Case 2** below. (Note: In step 6, the non-emptiness of $|T + (A \vee B)|$ is guaranteed by axiom (+4) solely because $A \vee B$ is consistent. it does not require T to be consistent).

Case 2: T is consistent. Let $U, V \in \mathbf{S}$. Suppose for reductio that $U \nsubseteq V$ and $V \nsubseteq U$. This means there exist worlds $u \in U \setminus V$ and $v \in V \setminus U$.

1. Since $u \in U$, by (Relevance) there is some A such that $u \in |T + A|$. By (+2), $A \in T + A$, hence for all $m \in |T + A|$, $A \in m$—that is, $|T + A| \subseteq |A|$. So, $A \in u$. Since $u \in |A| \cap U$, the intersection is not empty, and by (Swallowing) we have $|T + A| \subseteq U$.
2. Since $u \in |T + A|$ and $u \notin V$, it follows that $|T + A| \nsubseteq V$.
3. By the contrapositive of **Swallowing** for V: If $|T + A| \nsubseteq V$, then $|A| \cap V = \emptyset$.
4. Analogously for v: There exists B such that $v \in |T + B| \subseteq V$. Since $v \notin U$, $|T + B| \nsubseteq U$. By Swallowing for U, $|B| \cap U = \emptyset$.
5. Consider now $A \vee B$.
   - $u \in |T + A| \subseteq |A| \subseteq |A \vee B|$. Since $u \in U$, $|A \vee B| \cap U \neq \emptyset$. By Swallowing, $|T + (A \vee B)| \subseteq U$.
   - $v \in |T + B| \subseteq |B| \subseteq |A \vee B|$. Since $v \in V$, $|A \vee B| \cap V \neq \emptyset$. By Swallowing, $|T + (A \vee B)| \subseteq V$.
   - Therefore, $|T + (A \vee B)| \subseteq U \cap V$.
6. By (+4), since T is consistent *ex hypothesi*, $T + (A \vee B)$ is consistent (unless $A \vee B$ is logically false, which is impossible since $u \ni A$), so $|T + (A \vee B)| \neq \emptyset$. Let $z \in |T + (A \vee B)|$.
   - $z \in U$ and $z \in V$.
   - Also, since $A \vee B \in T + (A \vee B)$ by (+2), we have $z \in |A \vee B|$, so $z \in |A|$ or $z \in |B|$.
   - If $z \in |A|$: since $z \in V$, $|A| \cap V \neq \emptyset$. Contradiction with (3).
   - If $z \in |B|$: since $z \in U$, $|B| \cap U \neq \emptyset$. Contradiction with (4).
7. Either way, we get a contradiction. So, our initial assumption was false. $\mathbf{S}$ is nested.

(S2) Let me show tha (1) $|T| \in \mathbf{S}$ and that, (2) for all $U \in \mathbf{S}$, $|T| \subseteq U$.

(1) To prove that $|T| \in \mathbf{S}$, distinguish two cases.

(1.1) If T is inconsistent, i.e. $T = F$, then $|T| = \emptyset$, and by construction of $\mathbf{S}$ for the inconsistent case, $\emptyset \in \mathbf{S}$. Moreover, $|T|$ is hence a $\subseteq$-minimum for $\mathbf{S}$ in this case, for $\emptyset \subseteq U$ for any set U.

So, suppose T is consistent.

1. (Relevance) Let me show that Relevance holds for $|T|$. Consider now an arbitrary $m \in |T|$ and an arbitrary tautology in L, which I will call $\top$. Since T is consistent *ex hypothesi*, $\neg\top \notin T$, hence by (+3) it follows that $T + \top = Cn(T \cup \{\top\})$. Since $\top \in L \subseteq T$, $T + \top = Cn(T) = T$, for T is a theory *ex hypothesi*. Therefore, it immediately follows that $m \in |T + \top| = |T|$.
2. (Swallowing) Consider an arbitrary $A \in F$ and suppose $|A| \cap |T| \neq \emptyset$. That is, there exists a possible world $m \in M_L$ such that $A \in m$ and $T \subseteq m$. Therefore, $\neg A \notin T$ (otherwise, $A, \neg A \in m$, contradicting the consistency of $m \in M_L$). By (+3) $T + A = Cn(T \cup \{A\})$. This implies that $|T + A| = |T| \cap |A|$ [see callout below]. Therefore, we have that $|T + A| = |T| \cap |A| \subseteq |T|$, proving that $|T|$ satisfies (Swallowing).

If $\neg A \notin T$, $|T + A| = |Cn(T \cup \{A\})| = |T| \cap |A|$ >

(Note: This assumes the case where $\neg A \notin T$, so $T + A = Cn(T \cup \{A\})$.)

($\subseteq$) Suppose $m \in M_L$ is such that $m \in |T + A|$. This means $Cn(T \cup \{A\}) \subseteq m$. Since $T \subseteq Cn(T \cup \{A\})$ and $A \in Cn(T \cup \{A\})$, it follows that $T \subseteq m$ and $A \in m$. Therefore, $m \in |T|$ and $m \in |A|$, so $m \in |T| \cap |A|$.

($\supseteq$) Suppose $m \in M_L$ is such that $m \in |T| \cap |A|$. From the latter, $T \subseteq m$ and $A \in m$, which implies $T \cup \{A\} \subseteq m$. By Monotonicity for Cn, $Cn(T \cup \{A\}) \subseteq Cn(m)$. Since m is a world (maximal consistent set), $Cn(m) = m$. Thus, $Cn(T \cup \{A\}) \subseteq m$, meaning $m \in |Cn(T \cup \{A\})|$. Since $\neg A \notin T$ *ex hypothesi*, $T + A = Cn(T \cup \{A\})$, hence $m \in |T + A|$.

Concluding, $|T + A| = |T| \cap |A|$.

Therefore, $|T| \in \mathbf{S}'$, and so $|T| \in \mathbf{S}$. Consider now any $U \in \mathbf{S}$, and let me show that $|T| \subseteq U$. First, note that either $U = M_L$ or $U \in \mathbf{S}'$, which implies that U is nonempty.

1. $U \in \mathbf{S}'$. Since $\top \in F$, $|\top| = M_L$ and U is nonempty, we have that $U \cap |\top| \neq \emptyset$. By (Swallowing), $|T + \top| \subseteq U$. Since T is consistent *ex hypothesi*, $\neg\top \notin T$, hence by (+3) it follows that $T + \top = Cn(T \cup \{\top\})$. Since $\top \in L \subseteq T$, $T + \top = Cn(T) = T$, for T is a theory *ex hypothesi*. Therefore, $|T + \top| = |T|$, and so $|T| \subseteq U$. Since U is arbitrary, $|T|$ is a $\subseteq$-minimum.
2. $U = M_L$. Obviously, $|T| \subseteq M_L$.

Thus, (S2) holds.

(S3) $M_L \in \mathbf{S}$ by construction.

(S4) We must prove: If $|A| \cap \bigcup \mathbf{S} \neq \emptyset$ (i.e., if A is consistent), then there exists a sphere $U \in \mathbf{S}$ such that:

1. $U \cap |A| \neq \emptyset$ (It intersects $|A|$).
2. For any $V \in \mathbf{S}$, if $V \cap |A| \neq \emptyset$, then $U \subseteq V$ (It is the smallest such sphere).

Assume $|A| \cap \bigcup \mathbf{S} \neq \emptyset$ (i.e., A is consistent). We construct the specific set U_A as defined by Grove.

$$U_A \quad := \quad \bigcup \{|T + B| : |A| \subseteq |B|\}$$

(In words: The union of all revisions by B where B is logically weaker than A).

1. **Prove (1)**. Naturally, $|A| \subseteq |A|$ holds. Therefore, $|T + A|$ is one of the sets included in the union U_A. Since $|A| \neq \emptyset$, $\neg A \notin L$ (note that $L \subseteq m$, for all $m \in M_L$). By postulate (+4), $T + A$ is consistent, so $|T + A| \neq \emptyset$. By postulate (+2), $A \in T + A$, and so $|T + A| \subseteq |A|$. Since $|T + A| \subseteq U_A$, $|T + A| \subseteq |A|$, and $|T + A| \neq \emptyset$, then

$$U_A \cap |A| \neq \emptyset.$$

1. **Prove (2a)**. We must show $U_A \in \mathbf{S}$, i.e. that it satisfies the two conditions for $\mathbf{S}'$: Relevance and Swallowing.
    1. (Relevance). By definition, U_A is a union of sets of the form $|T + B|$. Therefore, every world $u \in U_A$ belongs to some $|T + B|$.
    2. (Swallowing) We need to prove that, for all $C \in F$, if $|C| \cap U_A \neq \emptyset$, then $|T + C| \subseteq U_A$. Suppose $|C| \cap U_A \neq \emptyset$. By the definition of U_A, this implies that there is some $B \in F$ such that (i) $|A| \subseteq |B|$ and (ii) $|C| \cap |T + B| \neq \emptyset$.

       Consider the sentence $B \vee C$. Note that $|A| \subseteq |B| \subseteq |B \vee C|$. Therefore, by the definition of U_A, $|T + (B \vee C)|$ is part of the union U_A, so $|T + (B \vee C)| \subseteq U_A$. We now claim that $|C| \cap |T + (B \vee C)| \neq \emptyset$. Suppose for reductio that $|C| \cap |T + (B \vee C)| = \emptyset$. This implies that, for all worlds $m \in M_L$ such that $T + (B \vee C) \subseteq m$, $C \notin m$, and hence $C \notin T + (B \vee C)$. Since $T + (B \vee C)$ is a theory, $\neg C \in T + (B \vee C)$. Since $T + (B \vee C)$ is consistent by (+4), it cannot imply $\neg(B \vee C)$. Since it implies $\neg C$, it must be that $\neg B \notin T + (B \vee C)$ (otherwise it would imply both negations, and thus the negation of the disjunction). Since $\neg B \notin T + (B \vee C)$, we can apply (+7) and (+8) to establish:

       $$T + ((B \vee C) \wedge B) = Cn((T + (B \vee C)) \cup \{B\})$$

       Since $((B \vee C) \wedge B)$ is logically equivalent to B, the LHS is $T + B$. If $\neg C \in T + (B \vee C)$ (as assumed), then $\neg C$ must also be in the expansion on the RHS. Therefore, $\neg C \in T + B$. But this means $|C| \cap |T + B| = \emptyset$ (for any $m \in M_L$ is consistent), contradicting assumption (ii) above.

Therefore, $|C| \cap |T + (B \vee C)| \neq \emptyset$, which implies $\neg C \notin T + (B \vee C)$. We can now apply (+8):

$$Cn((T + (B \vee C)) \cup \{C\}) \subseteq T + ((B \vee C) \wedge C) = T + C$$

We now translate this syntactic inclusion into semantic set inclusion. Recall that if theory $X \subseteq Y$, then $|Y| \subseteq |X|$.

$$\begin{aligned} |T + C| &\subseteq |Cn((T + (B \vee C)) \cup \{C\})| \\ &= |T + (B \vee C)| \cap |C| \qquad [*] \\ &\subseteq |T + (B \vee C)| \end{aligned}$$

Since we established earlier that $|T + (B \vee C)| \subseteq U_A$, it follows by transitivity that $|T + C| \subseteq U_A$. Thus, Swallowing holds.

Step $[*]$ >

The step denoted by $[*]$ relies on a crucial relationship between *syntactic expansion* and *semantic intersection of truth-sets*. The following provides justification for why $|Cn(K \cup \{C\})| = |K| \cap |C|$ (where K is the theory $T + (B \vee C)$).

First, note that the following holds:

$$m \in |Cn(K \cup \{C\})| \iff K \subseteq m \text{ and } C \in m$$

($\implies$) obviously holds: since $K \subseteq Cn(K \cup \{C\})$ and $C \in Cn(K \cup \{C\})$, if $Cn(K \cup \{C\}) \subseteq m$, clearly $K \subseteq m$ and $C \in m$. ($\impliedby$) suppose now that $K \subseteq m$ and $C \in m$. Since $K \cup \{C\} \subseteq m$ *ex hypothesi*, by Monotonicity $Cn(K \cup \{C\}) \subseteq Cn(m) = m$, for m is a theory. Hence, $m \in |Cn(K \cup \{C\})|$.

Moreover, the following is clearly true:

$$K \subseteq m \text{ and } C \in m \iff m \in |K| \cap |C|$$

For

1. $K \subseteq m$ is the definition of $m \in |K|$.
2. $C \in m$ is the definition of $m \in |C|$.

Combining the two iffs, we get $m \in |Cn(K \cup \{C\})| \iff m \in |K| \cap |C|$, which means that:

$$|Cn(K \cup \{C\})| = |K| \cap |C|$$

N.b. Don't forget that $|\cdot|$ returns different value, depending on whether its argument is a formula $C \in F$ of a set of formulas $K \subseteq F$!

2. **Prove (2b)**. We need to prove U_A is the Smallest Sphere (Minimality). Let V be any sphere in $\mathbf{S}$ such that $V \cap |A| \neq \emptyset$. We must show $U_A \subseteq V$. Take any component $|T + B|$ that makes up U_A (where $|A| \subseteq |B|$). Since $V \cap |A| \neq \emptyset$ and $|A| \subseteq |B|$, it follows that $V \cap |B| \neq \emptyset$.

   Since V is a sphere in $\mathbf{S}$, it must satisfy the **Swallowing** condition. Because V intersects $|B|$, by (Swallowing) $|T + B| \subseteq V$. Since this holds for every B such that $|A| \subseteq |B|$, the entire union is contained in V:

$$U_A \subseteq V$$

We have constructed a set U_A that is a valid sphere in $\mathbf{S}$, intersects $|A|$, and is a subset of every other sphere intersecting $|A|$. Thus, Condition (S4) holds.

Part Two: $T + A = t(f_{\mathbf{S}}(A))$.

We need to verify that the system of spheres $\mathbf{S}$ we constructed actually generates the original revision function $+$. Specifically, we want to prove:

$$T + A = t(f_{\mathbf{S}}(A))$$

We will prove the stronger condition that the sets of worlds are identical:

$$|T + A| = |A| \cap c(A)$$

Case 1: A **is Inconsistent (**$\neg A \in L$**)**.

1. By Postulate **(+2)**, $A \in T + A$. If A is logically false, then $T + A$ contains a contradiction, so $T + A = F$ (the inconsistent theory). Thus, $|T + A| = \emptyset$.
2. On the sphere side, if A is inconsistent, $|A| = \emptyset$. Therefore, $f_{\mathbf{S}}(A) = |A| \cap c(A) = \emptyset \cap c(A) = \emptyset$.
3. The equality holds: $|T + A| = \emptyset = |A| \cap c(A)$.

Case 2: A **is Consistent (**$\neg A \notin L$**)**. We prove the equality by mutual inclusion.

Direction 1: $|T + A| \subseteq |A| \cap c(A)$

1. Since A is consistent, by Postulate **(+4)**, $T + A$ is consistent, so $|T + A| \neq \emptyset$. Also, $A \in T + A$ implies $|T + A| \subseteq |A|$.
2. Recall that $c(A)$ is the smallest sphere in $\mathbf{S}$ intersecting $|A|$. Since $|T + A| \subseteq |A|$ and $|T + A| \neq \emptyset$, we know $|A| \cap \bigcup \mathbf{S} \neq \emptyset$, so $c(A)$ exists.
3. Since $c(A)$ is a sphere in our system $\mathbf{S}$, it must satisfy the **Swallowing** condition, i.e. if $|A| \cap c(A) \neq \emptyset$, then $|T + A| \subseteq c(A)$.
4. Since $c(A)$ is the smallest sphere intersecting $|A|$, the antecedent is true. Therefore, $|T + A| \subseteq c(A)$.
5. Combining this with $|T + A| \subseteq |A|$, we get:

$$|T + A| \subseteq |A| \cap c(A)$$

Direction 2: $|A| \cap c(A) \subseteq |T+A|$. This relies on the specific sphere U_A constructed in Part One. Recall $U_A = \bigcup\{|T+B| : |A| \subseteq |B|\}$.

1. **Minimality:** In Part One, we proved that U_A is a sphere intersecting $|A|$ and that for any sphere V intersecting $|A|$, $U_A \subseteq V$. By definition, $c(A)$ is the smallest sphere intersecting $|A|$. Therefore, $c(A) \subseteq U_A$. *(Note: In fact, $c(A) = U_A$, but inclusion is sufficient here).*
2. **Intersecting:** Since $c(A) \subseteq U_A$, intersecting both sides with $|A|$ preserves the subset relation:

$$|A| \cap c(A) \subseteq |A| \cap U_A$$

3. **Collapsing the Union:** We now evaluate $|A| \cap U_A$.

$$\begin{aligned} |A| \cap U_A &= |A| \cap \bigcup\{|T+B| : |A| \subseteq |B|\} \\ &= \bigcup\{|A| \cap |T+B| : |A| \subseteq |B|\} \end{aligned}$$

Consider any term $|A| \cap |T+B|$ in this union (where $|A| \subseteq |B|$). If the intersection is empty, it contributes nothing. If the intersection is non-empty ($|A| \cap |T+B| \neq \emptyset$), then $\neg A \notin T+B$. We can apply postulates **(+7)** and **(+8)**:

$$\begin{aligned} (T+B)/A &= T+(B \wedge A) \\ &= T+A \qquad\qquad (\text{since } |A| \subseteq |B|, |A \wedge B| = |A| \cap |B| = |A|) \end{aligned}$$

In terms of worlds: $|T+B| \cap |A| = |T+A|$. Therefore, every non-empty component of the union is exactly equal to $|T+A|$. Thus:

$$|A| \cap U_A = |T+A|$$

4. **Conclusion:** Substituting this back into step 2:

$$|A| \cap c(A) \subseteq |T+A|$$

Concluding, since we have proved both $|T+A| \subseteq |A| \cap c(A)$ and $|A| \cap c(A) \subseteq |T+A|$, we have:

$$|T+A| = |A| \cap c(A)$$

Since $T+A$ is a theory, by Proposition 8 (Properties of t) (prop.2) applying the theory operator $t(\cdot)$ to both sides yields:

$$\begin{aligned} t(|T+A|) &= t(|A| \cap c(A)) \\ T+A &= t(f_{\mathbf{S}}(A)) \end{aligned}$$

□

5. Alternative Modeling

Grove presents an alternative method for modeling a belief revision operation $+$ using a relation $\leq$ on the set of formulas F. This modeling is advantageous because the axioms for revision do *not* uniquely determine the specific changes required; generally, there are multiple valid choices for $T+A$ upon revision by A. Consequently, it is natural to consider an ordering of sentences to facilitate decisions regarding which elements to remove from or add to T.

Definition 15 (Plausibility Relation ($\leq$)).

Consider the relation $\leq \subseteq F^2$ satisfying the following properties:

1. $\leq$ is connected.
2. $\leq$ is transitive.
3. If $A \to B \vee C \in L$, then $B \leq A$ or $C \leq A$.
4. A is $\leq$-minimal in F if and only if $\neg A \notin T$.
5. A is $\leq$-maximal in F if and only if $\neg A \in L$.

The expression $A \leq B$ is interpreted as "A is at least as plausible as B".

Observations regarding these axioms:

- **Axiom (3)** implies that if $A \to B$, then $B \leq A$. This follows from the equivalence of B with $B \vee B$, and the fact that "$(B \leq A) \vee (B \leq A)$" is equivalent to $B \leq A$.
- **Axiom (4)** assigns the minimal position to any sentence A that is *either* in T or consistent with T (assuming T is a consistent theory). Since revision $+$ based on $\leq$ functions by selecting the sentences that are "most plausible" given T, it is reasonable to assign a privileged position to such sentences. These instances do not require actual revision (removing information from T), but rather augmentation (adding information without loss).

To prove that this is indeed an alternative modeling of $+$, Grove shows that this relation is equivalent to a system of spheres $\mathbf{S}$ centered on T.

Lemma 16.

Consider a system of spheres $\mathbf{S}$ centered on $|T|$, and define the following relation on F:

$$A \leq_S B : \iff c(A) \subseteq c(B)$$

and let us stipulate that $A <_S B$ iff $c(A) \subset c(B)$. It follows that:

1. *$\leq_S$ satisfies axioms (1)-(5);*
2. *the following definition of revision is identical to the one determined by $\mathbf{S}$:*

$$T + A := \{B \in F : A \wedge B <_S A \wedge \neg B\}$$

Proof. **Part One**. Let me first prove that $\leq_S$ satisfies axioms (1)-(5) of Definition 15 (Plausibility Relation ($\leq$)).

- $(\leq 1) - (\leq 2)$. Connectivity follows from the fact that $\mathbf{S}$ is nested, while transitivity follows from the properties of $\subseteq$.
- (≤ 3). Suppose $A \to B \vee C \in L$. Since $A \to B \vee C$, we have that $|A| \subseteq |B \vee C| = |B| \cup |C|$ (if $m \in |A|$, since $A \to B \vee C \in L \subseteq m$ and m is a theory, $B \vee C \in m$, and since m is consistent either $B \in m$ or $C \in m$). We have to show that either $c(B) \subseteq c(A)$ or $c(C) \subseteq c(A)$.
  1. First, consider consider $c(B \vee C)$, namely the sphere in $\mathbf{S}$ that intersects $|B \vee C|$ and such that, for any other sphere U intersecting $|B \vee C|$, $c(B \vee C) \subseteq U$. Consider $c(A)$. By definition, $c(A) \cap |A| \neq \emptyset$, and since $|A| \subseteq |B \vee C|$, we have $c(A) \cap |B \vee C| \neq \emptyset$. By definition, it follows $c(B \vee C) \subseteq c(A)$.
  2. Now, we observe that $c(B \vee C) = c(B)$ or $c(B \vee C) = c(C)$. Note that $|B| \subseteq |B \vee C|$, so any sphere intersecting $|B|$ intersects $|B \vee C|$, implying $c(B \vee C) \subseteq c(B)$. Similarly $c(B \vee C) \subseteq c(C)$.
  3. Since $c(B \vee C)$ intersects $|B| \cup |C|$, it must intersect at least one of them. If it intersects $|B|$, then by definition of $c(B)$, $c(B) \subseteq c(B \vee C)$. Combined with the above, $c(B) = c(B \vee C) \subseteq c(A)$, so $B \leq_S A$. If it intersects $|C|$, then $c(C) = c(B \vee C) \subseteq c(A)$, so $C \leq_S A$.
- **Minimality (≤ 4):** A is minimal iff $c(A) \subseteq c(B)$ for all B. This holds iff $c(A)$ is the smallest sphere in $\mathbf{S}$, which is $|T|$ (by S2). $c(A) = |T|$ iff $|A| \cap |T| \neq \emptyset$, which is equivalent to $\neg A \notin T$.
- **Maximality (≤ 5):** A is maximal iff $c(B) \subseteq c(A)$ for all B. This holds iff $c(A)$ is the largest sphere, M_L (by S3). By convention, $c(A) = M_L$ iff $|A| = \emptyset$, i.e., $\neg A \in L$.

Part Two. Define $T + A$ as follows:

$$T + A := \{B \in F : A \wedge B <_S A \wedge \neg B\} \tag{†}$$

We show that this set is equal to $t(f_{\mathbf{S}}(A))$. ($\subseteq$) Consider any $B \in$ (†). By definition:

$$A \wedge B <_S A \wedge \neg B$$
$$c(A \wedge B) \subset c(A \wedge \neg B)$$

Suppose for reductio that $B \notin t(f_{\mathbf{S}}(A))$. This means there exists some world $m \in c(A) \cap |A|$ such that $B \notin m$. Since m is complete, $\neg B \in m$, and so $m \in |A \wedge \neg B|$. Since $c(A)$ intersects $|A \wedge \neg B|$ (at m), it follows that $c(A \wedge \neg B) \subseteq c(A)$. Also, note that by definition, $c(A \wedge B)$ intersects $|A \wedge B|$, and since $|A \wedge B| \subseteq |A|$, it intersects $|A|$. Thus $c(A) \subseteq c(A \wedge B)$. Combining these inclusions: $c(A \wedge \neg B) \subseteq c(A) \subseteq c(A \wedge B)$. However, *ex hypothesi*, $c(A \wedge B) \subset c(A \wedge \neg B)$. This implies a contradiction ($X \subseteq Y$ and $Y \subset X$ is impossible). Therefore, $B \in t(f_{\mathbf{S}}(A))$.

($\supseteq$) Let $B \in t(f_{\mathbf{S}}(A))$. By definition, $B \in m$ for all $m \in f_{\mathbf{S}}(A) = c(A) \cap |A|$, and so $f_{\mathbf{S}}(A) = c(A) \cap |A| \subseteq |B|$. This implies that $c(A)$ contains no worlds where A is true and B is false. Equivalently:

$$c(A) \cap |A \wedge \neg B| = \emptyset$$

Suppose for reductio that $B \notin (\dagger)$. This means $A \wedge B \not<_S A \wedge \neg B$. Since $\leq_S$ is connected, this implies $A \wedge \neg B \leq_S A \wedge B$, or:

$$c(A \wedge \neg B) \subseteq c(A \wedge B)$$

Now we observe the relationship with $c(A)$. Since $c(A)$ intersects $|A|$, it must intersect either $|A \wedge B|$ or $|A \wedge \neg B|$ (that is, since there is at least one $m \in |A| \cap c(A)$, either $B \in m$ of $\neg B \in m$). We established above that $c(A) \cap |A \wedge \neg B| = \emptyset$. Therefore, $c(A)$ *must* intersect $|A \wedge B|$. Since $c(A \wedge B)$ is the smallest sphere intersecting $|A \wedge B|$, it follows that $c(A \wedge B) \subseteq c(A)$. Combining these facts:

$$c(A) \subseteq c(A \wedge \neg B) \subseteq c(A \wedge B) \subseteq c(A)$$

This chain implies equality: $c(A) = c(A \wedge \neg B)$. However, by definition, $c(A \wedge \neg B)$ must intersect $|A \wedge \neg B|$. If they are equal, $c(A)$ must intersect $|A \wedge \neg B|$. But we established at the start that $c(A) \cap |A \wedge \neg B| = \emptyset$: contradiction.

Therefore:

$$\{B \in F : A \wedge B <_S A \wedge \neg B\} \quad = \quad t(f_{\mathbf{S}}(A))$$

completing our proof. □

Corollary 17.

The belief revision operator $+$ defined via the relation $\leq_S$ satisfies the AGM postulates (+1)–(+8).

Proof. This result is an immediate consequence of Lemma Lemma 16 and Theorem Theorem 13 (Soundness).

1. Lemma Lemma 16 establishes that the revision operator defined by the relation (i.e., $\{B : A \wedge B <_S A \wedge \neg B\}$) is identical to the operator defined by the system of spheres (i.e., $t(f_{\mathbf{S}}(A))$).
2. Theorem Theorem 13 (Soundness) establishes that the operator defined by the system of spheres satisfies postulates (+1)–(+8).

Therefore, by transitivity, the operator defined via $\leq_S$ satisfies the postulates. □

Theorem 18 (Representation by Ordering on Formulas).

Let $+ : \mathcal{T} \times F \to \mathcal{T}$ be any function satisfying (+1)-(+8). Then for any (fixed) theory T there is a relation $<$ on F such that, for all $A \in F$:

$$T + A = \{B \in F : (A \wedge B) < (A \wedge \neg B)\}$$

Proof. This result follows from combining the representation of $+$ by spheres (Theorem 2) with the equivalence between spheres and formula orderings.

Since $+$ satisfies postulates (+1)-(+8), by **Theorem 2** Theorem 14 (Completeness), there exists a system of spheres $\mathbf{S}$ centered on $|T|$ such that for all $A \in F$, $T + A = t(f_{\mathbf{S}}(A))$. Given this system $\mathbf{S}$, we define the relation $\leq$ on F as:

$$A \leq B \iff c(A) \subseteq c(B)$$

and $A < B$ iff $c(A) \subset c(B)$. By Lemma Lemma 16, the revision operator determined by this relation is identical to the one determined by the system of spheres $\mathbf{S}$. Specifically:

$$t(f_{\mathbf{S}}(A)) = \{B \in F : (A \wedge B) < (A \wedge \neg B)\}$$

Substituting the identity from step 1 into step 3, we obtain $T + A = \{B \in F : (A \wedge B) < (A \wedge \neg B)\}$. □

Theorem 19 (Soundness of Ordering-Based Revision)**.**

Let $\leq$ be any relation on F satisfying postulates (≤ 1) to (≤ 5). Then the revision operation defined by:

$$T + A = \{B \in F : (A \wedge B) < (A \wedge \neg B)\}$$

satisfies the AGM axioms (+1) to (+8).

Proof. **Strategy:** We prove this by constructing a system of spheres $\mathbf{S}$ from the relation $\leq$ and showing they determine the same revision. Since sphere-based revision satisfies the axioms Theorem 13 (Soundness), the relation-based revision must as well.

□